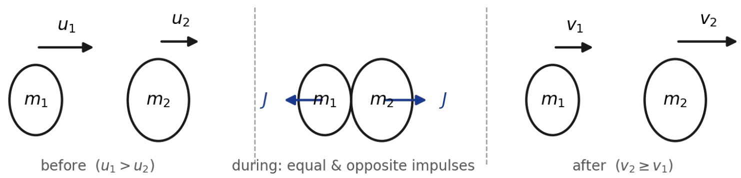
Head-on Collision — Formula Sheet

Class XI Physics · Centre of Mass & Collisions · valid for any coefficient of restitution $0 \le e \le 1$



Convention. Velocities are components along the line of impact, positive to the right. m_1 is the body *behind*, so $u_1 > u_2$ and the bodies approach. During the brief contact each body receives an impulse of the same magnitude J: backward on m_1, forward on m_2.

Impulse: $\mathbf{J = \int F\,dt = F_{av}\,\Delta t}$, Δt = duration of contact. Approach speed $\mathbf{v_{app} = u_1 - u_2}$; separation speed $\mathbf{v_{sep} = v_2 - v_1}$.

The six results

①	$v_{sep} = e\,v_{app} \Rightarrow \mathbf{v_2 - v_1 = e\,(u_1 - u_2)}$	Newton's law of restitution. Defines e; e = 1 perfectly elastic, e = 0 perfectly inelastic (bodies move together).
②	$J = \mu\,(v_{app} + v_{sep}) = \boldsymbol{\mu\,v_{app}\,(1 + e)}$	Impulse on either body. Follows from ①, ④, ⑤: subtracting ④ from ⑤ gives $v_2 - v_1 = -v_{app} + J/\mu$.
③	$\boldsymbol{\mu = \dfrac{m_1 m_2}{m_1 + m_2}}$ or $\dfrac{1}{\mu} = \dfrac{1}{m_1} + \dfrac{1}{m_2}$	Reduced mass. Always smaller than the lighter body; equals m/2 for equal masses m; ≈ lighter mass when the other is very heavy.
④	$p_{1f} = p_{1i} - J \Rightarrow \mathbf{v_1 = u_1 - J/m_1}$	Impulse–momentum for m_1 (impulse opposes its motion).
⑤	$p_{2f} = p_{2i} + J \Rightarrow \mathbf{v_2 = u_2 + J/m_2}$	Impulse–momentum for m_2. Adding m_1×④ and m_2×⑤ recovers momentum conservation: J cancels.
⑥	$\Delta K = \tfrac{1}{2}\,\mu\,(v_{app}^2 - v_{sep}^2) = \boldsymbol{\tfrac{1}{2}\,\mu\,v_{app}^2\,(1 - e^2)}$	Loss of kinetic energy. Only the KE of relative motion, $\tfrac{1}{2}\,\mu\,v_{rel}^2$, can change; the KE of the centre of mass is untouched.

Handy corollaries (all follow from ① – ⑥)

7	$v_1 = \dfrac{(m_1 - e\,m_2)\,u_1 + (1 + e)\,m_2\,u_2}{m_1 + m_2}$ $v_2 = \dfrac{(m_2 - e\,m_1)\,u_2 + (1 + e)\,m_1\,u_1}{m_1 + m_2}$	Closed form: put ② into ④ and ⑤. Use when a direct answer is asked; use ④/⑤ when impulse or force is asked.
8	$v_{cm} = \dfrac{m_1 u_1 + m_2 u_2}{m_1 + m_2}$ unchanged, for every e	No external impulse along the line of impact. For e = 0 both bodies end up moving at v_{cm}.
9	$\Delta K = \tfrac{1}{2}\,J\,v_{app}\,(1 - e)$	Same as ⑥, written with the impulse — handy when J is already known.
10	KE fraction given to m_2 (at rest, e = 1) = $\dfrac{4\,m_1\,m_2}{(m_1 + m_2)^2}$	Maximum (100 %) for equal masses — the basis of choosing light nuclei as neutron moderators.

Special cases to recognise on sight

- **e = 1 (elastic):** $\Delta K = 0$; $v_{sep} = v_{app}$; $J = 2\mu\,v_{app}$. Equal masses ⇒ velocities are *exchanged*. Heavy on light at rest ⇒ light body leaves at ≈ 2u; light on heavy at rest ⇒ rebounds at ≈ u.
- **e = 0 (perfectly inelastic):** $v_1 = v_2 = v_{cm}$; $J = \mu\,v_{app}$; $\Delta K = \tfrac{1}{2}\,\mu\,v_{app}^2$ — the *maximum* possible loss for the given system.
- **m_1 stops dead** after hitting m_2 at rest only if $e = m_1/m_2$ (so it needs $m_1 \le m_2$).
- **Ball on fixed floor** ($m_2 \to \infty$): rebound speed = e × impact speed; rebound height = e^2 × drop height; total distance travelled before stopping $= h\,(1 + e^2)/(1 - e^2)$.

Head-on Collisions — Answer Book

Answer key *[full working follows; qualitative Q20–22 answered in words]*

Q1 B
Q2 B
Q3 C
Q4 C
Q5 B
Q6 A
Q7 B
Q8 B
Q9 C
Q10 A
Q11 B, C
Q12 A, B, D
Q13 A, B, C
Q14 A, C, D
Q15 $v_0/2$; $v_0\sqrt{(m/2k)}$
Q16 0.8 m ; 99 %
Q17 −1 & 1.5 m/s ; 12.5 J ; 16.7 J
Q18 $e = 1/4$; 45 J
Q19 $4v/3$; $mv^2/3$

Starting kit — the six formula-sheet results

① $v_2 - v_1 = e(u_1 - u_2) = e\, v_{app}$ ② $J = \mu\, v_{app}(1 + e)$ ③ $\mu = m_1 m_2/(m_1+m_2)$
④ $v_1 = u_1 - J/m_1$ ⑤ $v_2 = u_2 + J/m_2$ ⑥ $\Delta K = ½\, \mu\, v_{app}^2\, (1 - e^2)$ [m_1 is the body behind; $v_{app} = u_1 - u_2$]
Routine for every numerical: write v_{app} and μ ③ → get J from ② → velocities from ④ ⑤ → loss from ⑥.
Circled numbers below show which result each line starts from.

Detailed solutions

Q1. Ans: (B) $v/u = -1/2$

$v_{app} = u$, $e = 1$. ③ $\mu = m(3m)/(4m) = 3m/4$

② $J = \mu\, v_{app}(1 + e) = (3m/4)(u)(2) = 3mu/2$

④ $v_1 = u - J/m = u - 3u/2 = -u/2 \Rightarrow v/u = -1/2$ (rebounds) [⑤ gives 3m: $0 + J/3m = u/2$; check p: $-mu/2 + 3mu/2 = mu$ ✓]

Q2. Ans: (B) 1.2 kg

$e = 1$, $v_{app} = u$. ② $J = 2\mu u$. ④ $v_1 = u - 2\mu u/m_1 = u/4 \Rightarrow 2\mu/m_1 = 3/4 \Rightarrow \mu = 3m_1/8 = 0.75$ kg

③ $1/\mu = 1/m_1 + 1/m_2 \Rightarrow 1/m_2 = 4/3 - 1/2 = 5/6 \Rightarrow m_2 = 6/5 = 1.2$ kg [$1/0.75 = 4/3$ — reciprocal, not division]

Q3. Ans: (C) $\sqrt{2}\, v_0$

Equal masses: ③ $\mu = m/2$; $v_{app} = v_0$. KE rises by 50 %, so the "loss" is negative: $\Delta K = -\tfrac{1}{2} \cdot \tfrac{1}{2} m v_0^2$ [an explosive (super-elastic) collision, $e > 1$]

⑥ $\tfrac{1}{2}(m/2) v_0^2 (1 - e^2) = -\tfrac{1}{4} m v_0^2 \Rightarrow 1 - e^2 = -1 \Rightarrow e^2 = 2$

① $v_{sep} = e\, v_{app} = \sqrt{2}\, v_0$ [relative velocity after = separation speed]

Q4. Ans: (C) 0.67 J

Move together $\Rightarrow e = 0$; $v_{app} = 2$ m/s. ③ $\mu = (0.5 \times 1)/1.5 = 1/3$ kg

⑥ $\Delta K = \tfrac{1}{2} \mu v_{app}^2 (1 - 0) = \tfrac{1}{2} \times (1/3) \times 4 = 2/3$ J ≈ 0.67 J [$1/1.5 = 2/3$, so $0.5 \times 2/3 = 1/3$]

Q5. Ans: (B) 48/169

$e = 1$, $v_{app} = u$ (neutron speed). ③ $\mu = (1 \times 12)/13 = 12/13\, u$

② $J = \mu u (2) = 24u/13$. ④ $v_1 = u - J/1 = u - 24u/13 = -11u/13$ (neutron bounces back)

KE retained $= (11/13)^2 = 121/169 \Rightarrow$ fraction lost $= 1 - 121/169 = 48/169 \approx 0.28$ [$13^2 = 169$; $11^2 = 121$; $169 - 121 = 48$]

Q6. Ans: (A) 0, 0, v

Equal masses, $e = 1$: ③ $\mu = m/2$; ② $J = (m/2)(v)(2) = mv$

A on B: ④ $v_A = v - mv/m = 0$; ⑤ $v_B = 0 + mv/m = v$ — velocities exchanged. [$J = mv$ = whole momentum handed over]

B on C: same again $\Rightarrow v_B = 0$, $v_C = v$. Finally A = 0, B = 0, C = v.

Q7. Ans: (B) 1 : 2

③ $\mu = m/2$; $v_{app} = u$; $e = 1/3$. ② $J = (m/2)(u)(1 + 1/3) = (m/2)(4u/3) = 2mu/3$

④ $v_1 = u - (2mu/3)/m = u/3$; ⑤ $v_2 = 0 + (2mu/3)/m = 2u/3 \Rightarrow v_1 : v_2 = 1 : 2$ [① check: v_sep = $u/3 = e{\cdot}u$ ✓]

Q8. Ans: (B) 5h/3

Floor is $m_2 \to \infty$ $(u_2 = v_2 = 0)$. ① gives rebound speed = $e \times$ impact speed. [with $v^2 = 2gh$, height scales as e^2]

$h_1 = e^2h = h/4 \Rightarrow e = 1/2$.

Distance = $h + 2e^2h + 2e^4h + \ldots = h + 2e^2h/(1 - e^2) = h(1 + e^2)/(1 - e^2) = h\ (5/4)/(3/4) = 5h/3$ [GP, ratio $e^2 = 1/4$]

Q9. Ans: (C) mv

③ $\mu = m(2m)/3m = 2m/3$; $v_{app} = v$; $e = 1/2$.

② $J = \mu\, v_{app}(1 + e) = (2m/3)(v)(3/2) = mv$ — the impulse asked for, in one line. [④: $v_1 = v - mv/m = 0$, block m stops dead; ⑤: $v_2 = mv/2m = v/2$]

Q10. Ans: (A)

Statement 2 is the momentum law (no external force) — true for every e.

⑥ with $e = 0$ gives the largest possible loss, $\Delta K_{max} = ½\, \mu\, v_{app}^2$. What remains is $K_i - \Delta K_{max} = ½(m_1+m_2)v_{cm}^2 = p^2/2(m_1+m_2)$.

Same direction $\Rightarrow p \neq 0 \Rightarrow$ this remainder is > 0: all KE cannot be lost. Statement 2 explains 1 $\Rightarrow$ (A) [only when p_total = 0 can all KE vanish]

Q11. Ans: (B), (C)

A $\not\Rightarrow$ B: two equal particles moving opposite ways: $p = 0$ but $K > 0$. [KE is a sum of positive terms]

B $\Rightarrow$ A: $K = 0$ forces every particle to be at rest $\Rightarrow p = 0$.

So B implies A, A does not imply B.

Q12. Ans: (A), (B), (D)

$m \ll M \Rightarrow$ ③ $\mu = mM/(M + m) \approx m$. $e = 1$, $v_{app} = u \Rightarrow$ ② $J \approx 2mu$.

④ $v_1 = u - 2mu/M \approx u$ (A ✓); ⑤ $v_2 = 0 + 2mu/m = 2u$ (B ✓, C ✗) [golf club on ball: ball leaves at ≈ 2u]

Light projectile on heavy target: $\mu \approx m$ again, $J \approx 2mu$, ④ $v_1 = u - 2mu/m = -u \Rightarrow$ rebounds at the same speed (D ✓) [like a ball off a wall]

Q13. Ans: (A), (B), (C)

P: ④ $v_1 = u_1 - J/m_1 \Rightarrow -2 = 6 - J/1 \Rightarrow J = 8$ N s; impulse on Q is +8 N s (C ✓)

Q: ⑤ $v_2 = 0 + J/m_Q \Rightarrow 4 = 8/m_Q \Rightarrow m_Q = 2$ kg (A ✓)

① $e = (4 - (-2))/(6 - 0) = 6/6 = 1 \Rightarrow$ perfectly elastic (B ✓)

⑥ $\Delta K = ½\, \mu\, v_{app}^2\ (1 - 1) = 0 \Rightarrow$ no loss (D ✗) [check: 18 J before, 2 + 16 = 18 J after]

Q14. Ans: (A), (C), (D)

$u_2 = 0 \Rightarrow v_{app} = u$. ② $J = \mu(1 + e)u$; ④ $v_1 = u - \mu(1 + e)u/m_1 = u[1 - m_2(1 + e)/(m_1+m_2)] = u(m_1 - e\, m_2)/(m_1+m_2)$

$v_1 = 0 \Leftrightarrow e = m_1/m_2$ (A ✓); $e \leq 1 \Rightarrow$ needs $m_1 \leq m_2$, so a heavier projectile never stops (B ✗) [heavier body always keeps moving forward]

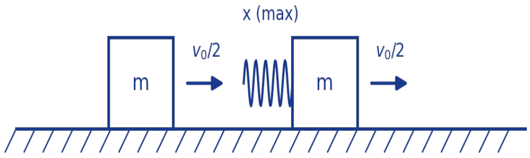
(C) is ⑥ word for word ✓. No external force $\Rightarrow V_{cm}$ unchanged for any e (D ✓)

Q15. Ans: (a) $v_0/2$ each (b) $x = v_0 \sqrt{(m/2k)}$

At maximum compression the blocks have the same velocity, i.e. $v_{sep} = 0$ — momentarily the $e = 0$ state of ①. [spring keeps compressing while the left block is faster]

③ $\mu = m/2$; $v_{app} = v_0$; ② $J = (m/2)(v_0)(1 + 0) = mv_0/2$

④ $v_1 = v_0 - (mv_0/2)/m = v_0/2$; ⑤ $v_2 = 0 + v_0/2$ — both at $v_0/2$ (a)



(b) The KE "lost" at this instant is stored in the spring: ⑥ with $e = 0$: $\frac{1}{2} \mu v_0^2 = \frac{1}{4} m v_0^2 = \frac{1}{2} k x^2$

$\Rightarrow x^2 = m v_0^2/2k \Rightarrow x = v_0 \sqrt{(m/2k)}$ [exactly half the KE is in the spring; it is returned later ($e = 1$ overall)]

Q16. Ans: (a) h = 0.8 m (b) 99 %

Embeds ⇒ $e = 0$; $v_{app} = 400$ m/s. ③ $\mu = (0.02 \times 1.98)/2.00 = 0.0198$ kg

② $J = \mu v_{app}(1 + 0) = 0.0198 \times 400 = 7.92$ N s [$0.02 \times 1.98 = 0.0396$; $\div 2 = 0.0198$]

⑤ $V = 0 + J/M = 7.92/1.98 = 4$ m/s [or ④: $400 - 7.92/0.02 = 400 - 396 = 4$ ✓]

Swing: $\frac{1}{2}(2)V^2 = (2)g h \Rightarrow h = V^2/2g = 16/20 = 0.8$ m

⑥ $\Delta K = \frac{1}{2} \mu v_{app}^2 = \frac{1}{2} \times 0.0198 \times 160000 = 1584$ J; $K_{bullet} = \frac{1}{2} \times 0.02 \times 160000 = 1600$ J ⇒ $1584/1600 = 99$ % lost [$1584/1600 = 1 - 16/1600 = 1 - 1/100$]

Q17. Ans: (a) −1 m/s, +1.5 m/s (b) 12.5 J (c) 16.7 J

Right +. $u_1 = +4$ (2 kg), $u_2 = -1$ (4 kg) ⇒ $v_{app} = 4 - (-1) = 5$ m/s. ③ $\mu = 8/6 = 4/3$ kg

② $J = \mu v_{app}(1 + e) = (4/3)(5)(1.5) = 10$ N s

④ $v_1 = 4 - 10/2 = -1$ m/s (2 kg bounces back); ⑤ $v_2 = -1 + 10/4 = +1.5$ m/s [① check: $1.5 - (-1) = 2.5 = 0.5 \times 5$ ✓]

(b) ⑥ $\Delta K = \frac{1}{2} (4/3)(25)(1 - 0.25) = \frac{1}{2} (4/3)(25)(3/4) = 12.5$ J [$(4/3)(3/4) = 1$, so it is just $25/2$]

(c) $e = 0$: ⑥ $\Delta K_{max} = \frac{1}{2} (4/3)(25) = 50/3 \approx 16.7$ J [both then move at $v_{cm} = 4/6 = 0.67$ m/s]

Q18. Ans: $e = 1/4$, $\Delta K = 45$ J

⑤ $v_2 = 0 + J/m_2 \Rightarrow 5 = J/2 \Rightarrow J = 10$ N s. ④ $v_1 = 12 - 10/1 = 2$ m/s

① $e = (5 - 2)/(12 - 0) = 3/12 = 1/4$

③ $\mu = 2/3$; ⑥ $\Delta K = \frac{1}{2} (2/3)(144)(1 - 1/16) = (1/3)(144)(15/16) = 48 \times 15/16 = 45$ J [$144/16 = 9$; $9 \times 15 = 135$; $135/3 = 45$]

Q19. Ans: $V = 4v/3$, $\Delta K = mv^2/3$

Sticks ⇒ $e = 0$; $v_{app} = 2v - v = v$. ③ $\mu = m(2m)/3m = 2m/3$

② $J = \mu v_{app} = 2mv/3$. ④ $V = 2v - (2mv/3)/m = 2v - 2v/3 = 4v/3$ [⑤: $v + (2mv/3)/2m = v + v/3 = 4v/3$ ✓ same]

⑥ $\Delta K = \frac{1}{2} \mu v_{app}^2 = \frac{1}{2} (2m/3) v^2 = mv^2/3$

Q20. Ans: Both momentum and KE must be conserved (steel balls ≈ elastic)

Let n balls leave with speed w each. Momentum: $n\,m\,w = m\,v$. KE: $n(\tfrac{1}{2}\,m\,w^2) = \tfrac{1}{2}\,m\,v^2$.

Dividing: $w = v$, and then $n = 1$. Only one ball, with the full speed, satisfies both.

Two balls at $v/2$: momentum $= mv$ ✓ but KE $= 2 \times \tfrac{1}{2}\,m\,(v/2)^2 = mv^2/4 \neq \tfrac{1}{2}\,mv^2$ ✗ [half the KE would have to vanish — impossible with $e = 1$ in ⑥]

(Each intermediate ball just exchanges velocity with the next — Q6.)

Q21. Ans: see below

(a) ① e = (relative speed of separation)/(relative speed of approach) $= (v_2 - v_1)/(u_1 - u_2)$, along the line of impact.

$e = 1$: perfectly elastic (⑥ gives $\Delta K = 0$); $e = 0$: perfectly inelastic (bodies stick, ⑥ maximum); $0 < e < 1$: partially elastic. [Newton's experimental law; depends on materials]

(b) $e = 1$, $u_2 = 0$: ② $J = 2\mu u$; ⑤ $v_2 = 2\mu u/m_2 = 2m_1u/(m_1 + m_2)$

$K_2/K_1 = [\tfrac{1}{2}\,m_2\,v_2^2]/[\tfrac{1}{2}\,m_1\,u^2] = 4m_1m_2/(m_1+m_2)^2$

This is 1 (100 %) when $m_1 = m_2$ and falls off rapidly for very unequal masses. [$(\sqrt{m_1} - \sqrt{m_2})^2 \geq 0 \Rightarrow$ fraction ≤ 1]

Moderator: neutron (1 u) must be slowed. Deuterium (2 u): $4{\cdot}2/9 = 0.89$; carbon (12 u): 0.28; lead (208 u): $4{\cdot}208/209^2 \approx 0.02$ per collision.

Light nuclei of mass close to the neutron's take away most of its KE — so D_2O or graphite, not lead. [D_2O rather than H_2O because ordinary H absorbs neutrons]

Q22. Ans: see below

During contact the bodies exert equal and opposite internal forces F and $-F$ (Newton III). Their impulses $\int F\,dt$ and $-\int F\,dt$ cancel exactly ⇒ total momentum is unchanged, whatever the nature of the forces — this is why ④ and ⑤ carry the <u>same</u> J with opposite signs. [vector sum of impulses cancels]

Work, however, is force × displacement of its point of application. The two contact faces move through <u>different</u> distances while deforming, so $W_1 + W_2 = -\int F\,d(x_1 - x_2) \neq 0$ in general. This net negative work removes KE — the amount is exactly ⑥. [scalar work does not cancel]

The "lost" KE becomes heat (internal energy), sound, permanent deformation, or is stored as elastic PE. In an elastic collision, during compression KE → elastic PE of the deformed bodies; at maximum compression both move with V_{cm} (the $e = 0$ state of ①) and KE is at its minimum $\tfrac{1}{2}(m_1+m_2)V_{cm}^2$. During restitution the stored PE is fully returned, so $K_{final} = K_{initial}$. [partially elastic: only part of the PE is returned → $e < 1$]

Working without a calculator

1. Fractions first, decimals last: keep 2/3, 5/3, 48/169 as fractions; decimalise only at the end.
2. Division → multiplication by the reciprocal: 1/1.5 = 2/3. For 48/169 use $169 = 13^2$: 48/13 = 3.69 (1/13 = 0.0769), then 3.69/13 = 0.284.
3. Percent loss in Q16: 1584/1600 = 1 − 16/1600 = 1 − 1/100 = 0.99 → 99 %. Never divide 1584 by 1600 long-hand.
4. Powers of ten separately: $\frac{1}{2} \times 0.02 \times 160000 = \frac{1}{2} \times 2 \times 16 \times 10^{-2} \times 10^4 = 16 \times 10^2 = 1600$.
5. Squares to memorise: $13^2 = 169$, $12^2 = 144$, $15^2 = 225$, $209^2 \approx 4.4 \times 10^4$ (so 832/43700 ≈ 0.019).
6. Reduced mass by inspection: $\mu = m_1m_2/(m_1+m_2)$ → (2·4)/6 = 4/3; (0.5·1)/1.5 = 1/3; for 1 kg & 2 kg it is 2/3.
7. Closed form (② into ④, ⑤) for m_2 at rest: $v_1 = (m_1 - e\, m_2)u/(m_1+m_2)$, $v_2 = (1+e)m_1u/(m_1+m_2)$. Equal masses: $v_1 = (1-e)u/2$, $v_2 = (1+e)u/2$.
8. KE-loss shortcut ⑥: $\Delta K = \frac{1}{2}\, \mu\, (1 - e^2)\, v_{app}^2$ — one line, no need to compute K before and after (but do it once as a check).
9. Surds: $\sqrt{2} = 1.414$, $\sqrt{3} = 1.732$, $1/\sqrt{2} = 0.707$; keep $\sqrt{(m/2k)}$ symbolic — the examiner wants the form, not a number.