

Motion in a Vertical Circle

Class XI Physics · Work, Energy & Circular Motion · Class notes (Zoom session, 16 September 2026)

Covered today: inner track — the four cases of v_0 · why N can never vanish in the lower half · outer track and $\cos^{-1}(2/3)$ · loop from a height · channel track · conical pendulum · a pendulum released from an angle (tension and acceleration of the bob)

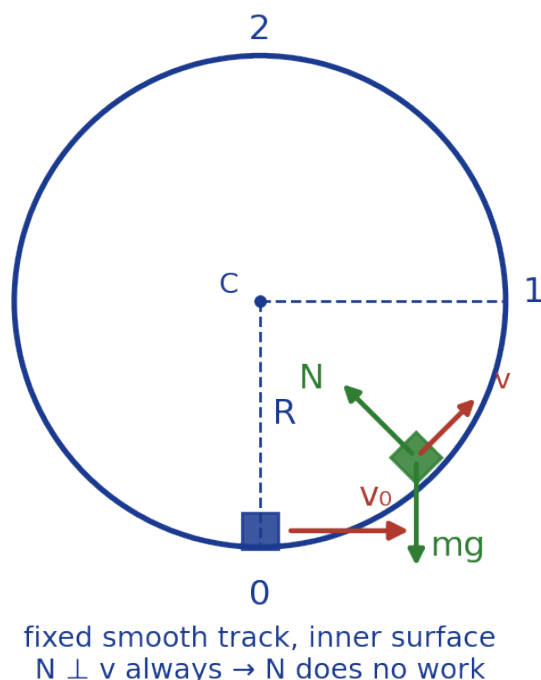
The only two tools used in this whole class

Energy equation — between two points, when only gravity does work: $\frac{1}{2}mv^2 + mgh$ is the same at both.

Force equation — at one point, along the radius: (net inward force) = mv^2/R . A body leaves a track when $N = 0$; a string goes slack when $T = 0$.

1. The set-up: a mass on the inside of a fixed vertical circular track

A circle of radius R in a vertical plane is a *trajectory* or *track* — think of the rim of a cycle tyre. A small mass can run along the **inside** of the rim (inner track) or slide over the **outside** (outer track). We do the inner one first. The track is **fixed** and **smooth**.



Mass m given speed v_0 at the lowest point 0. Points 1 (level with the centre) and 2 (top) are marked. At any point the forces are N (along the radius, towards C) and mg .

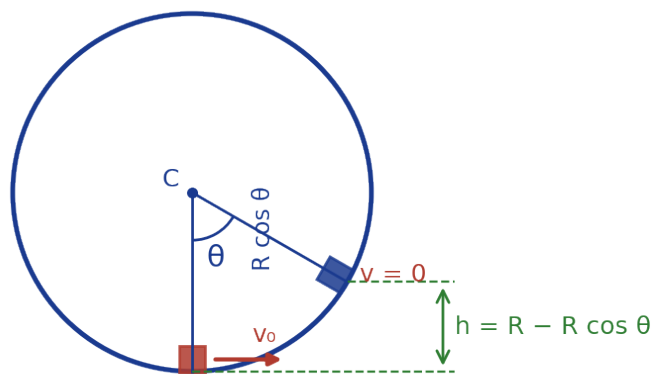
Why can we conserve mechanical energy? Friction is zero. N is not a conservative force — but it does **no work**, because it is always perpendicular to the displacement. So the only force doing work is gravity, which is conservative. Hence **total mechanical energy is conserved** between any two points.

Case 1 — v_0 for the mass to exactly reach point 1 and slide back

"Exactly reaching 1 and sliding back" means the speed at 1 is zero. TME at 0 = TME at 1:

$$\frac{1}{2} m v_0^2 + 0 = 0 + mgR \quad \Rightarrow \quad v_0 = \sqrt{2gR}$$

Case 2 — $v_0 < \sqrt{2gR}$: how high does it go?



With $v_0 < \sqrt{2gR}$ the mass rises to some angle θ , stops momentarily, and slides back. Its height above 0 is $h = R - R \cos \theta$.

Example: $v_0 = \sqrt{gR}$. Energy between 0 and the highest point (where $v = 0$):

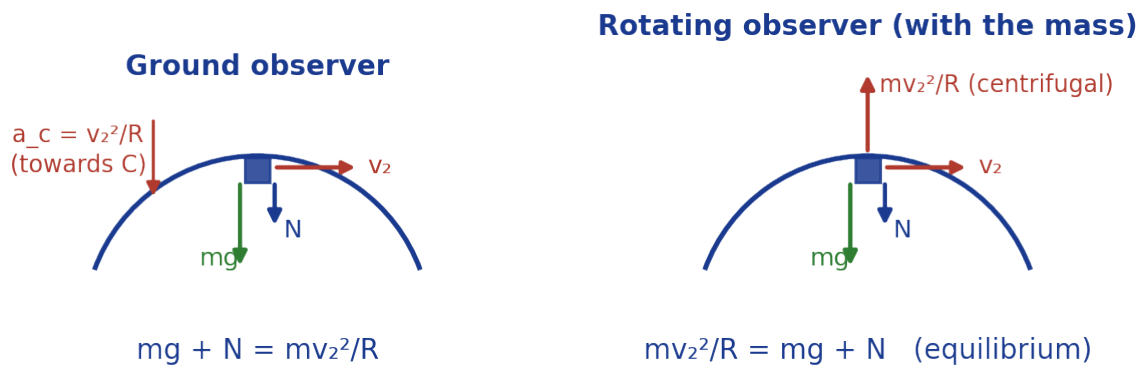
$$\frac{1}{2} m v_0^2 = mg h_{\max} \Rightarrow h_{\max} = v_0^2 / 2g = gR / 2g = R/2$$

$$R(1 - \cos \theta) = R/2 \Rightarrow \cos \theta = \frac{1}{2} \Rightarrow \theta = 60^\circ$$

In general, for any $v_0 < \sqrt{2gR}$: $\cos \theta = 1 - v_0^2 / (2gR)$. The mass then **oscillates** about the lowest point with angular amplitude θ .

Case 3 — v_0 for the mass to cross the top point 2

Can we take $v_2 = 0$ at the top? **No**. With zero speed it would fall like a dead weight and not move on the circle. There is a minimum v_2 . Look at the forces at the top from two points of view — both give the same condition:



At the top both mg and N point down (towards C). Left: ground observer — the net downward force is the centripetal force. Right: an observer moving with the mass — the outward centrifugal force mv_2^2/R balances $mg + N$.

Ground observer	Observer riding with the mass
The mass accelerates towards C at v_2^2/R . Net inward force: $mg + N = mv_2^2/R$.	The mass is in equilibrium. Outward centrifugal force $mv_2^2/R = mg + N$.
The track can only push, so $N \geq 0 \Rightarrow mv_2^2/R \geq mg$.	It does not fall as long as the outward force is at least mg : $mv_2^2/R \geq mg$.

$$v_2 \geq \sqrt{gR} \quad \text{at the top (} N = 0 \text{ exactly when } v_2 = \sqrt{gR}\text{)}$$

Now the energy equation between 0 and 2, with the minimum v_2 :

$$\frac{1}{2} m v_0^2 + 0 = \frac{1}{2} m (gR) + mg(2R) \Rightarrow v_0^2 = gR + 4gR \Rightarrow v_0 \geq \sqrt{5gR}$$

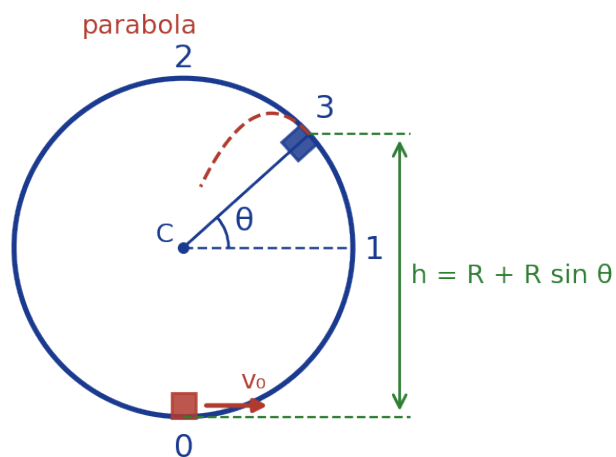
Sir's remark in class: in the village mela there is a wheel, and a motorcyclist goes round the inside — people say "arey, yeh toh bilkul ulta ho gaya upar!" Why does gravity not pull him down? Because gravity is being used up in providing the centripetal force. He is falling towards the centre all the time — just as the Moon is falling towards the Earth — but his distance from the centre does not change, because the falling acceleration equals the centripetal acceleration. Note that this is not uniform circular motion: his speed keeps changing.

Summary so far

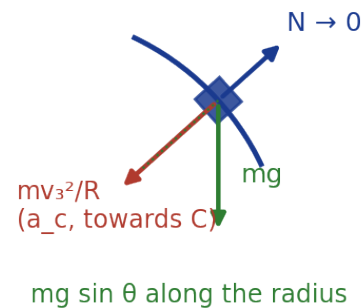
v_0	What happens
$v_0 < \sqrt{2gR}$	cannot reach 1; rises to $\theta < 90^\circ$ and slides back; oscillates with angular amplitude θ
$v_0 = \sqrt{2gR}$	reaches exactly 1 and slides back; oscillates with $\theta = 90^\circ$
$v_0 \geq \sqrt{5gR}$	successfully completes the vertical circle
$\sqrt{2gR} < v_0 < \sqrt{5gR}$	crosses 1 but cannot cross 2 — leaves the track somewhere in between (Case 4)

Case 4 — $\sqrt{2gR} < v_0 < \sqrt{5gR}$: where does it leave the track?

It crosses point 1, so it is in the upper half. It cannot reach 2. Its speed is not zero — so at some point 3 it **leaves the circular path and moves as a projectile** (a parabola). Find the angle θ of point 3 above the horizontal.



at the leaving point 3



Left: the leaving point 3 at angle θ above the horizontal through C; height gained from 0 is $h = R + R \sin \theta$; after 3 the path is a parabola. Right: forces at 3 — the radial component of mg is $mg \sin \theta$, N has just become zero.

Energy ($0 \rightarrow 3$), with the example $v_0 = \sqrt{4gR} = 2\sqrt{gR}$:

$$\frac{1}{2} m v_0^2 = \frac{1}{2} m v_3^2 + mg R(1 + \sin \theta) \quad \Rightarrow \quad v_3^2 = 4gR - 2gR - 2gR \sin \theta = 2gR(1 - \sin \theta)$$

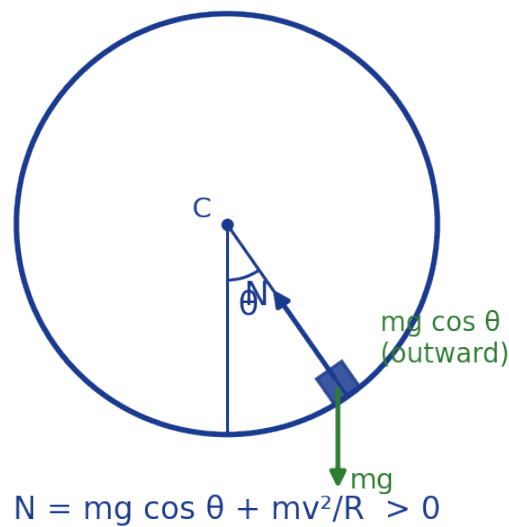
Force at 3, along the radius. What is special about 3? It is losing contact — so $N = 0$:

$$mg \sin \theta - N = m v_3^2 / R, \quad \text{with } N = 0 \quad \Rightarrow \quad g \sin \theta = v_3^2 / R = 2g(1 - \sin \theta)$$

$$3 \sin \theta = 2 \quad \Rightarrow \quad \sin \theta = 2/3 \quad (\text{for } v_0 = \sqrt{4gR})$$

For any v_0 in this range the same two steps give $\sin \theta = (v_0^2 - 2gR)/(3gR)$ and $v_3^2 = gR \sin \theta$. Check: $v_0^2 = 4gR$ gives $2/3$; $v_0^2 = 2gR$ gives $\theta = 0$ (leaves at 1); $v_0^2 = 5gR$ gives $\theta = 90^\circ$ (just reaches the top with $N = 0$). After 3 it is a projectile launched at speed v_3 at angle θ to the vertical — where it lands is a projectile-motion question. For JEE Main it is usually enough to know that the path is parabolic and that $N = 0$ is the leaving condition.

Can N be zero between 0 and 1?



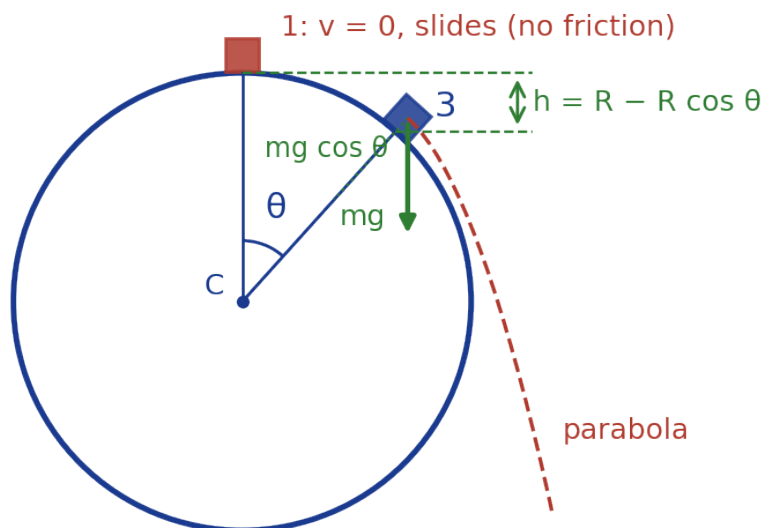
In the lower half, at angle θ from the bottom, the radial component of mg points outward (away from C). N must supply both that and the centripetal force.

$$N - mg \cos \theta = mv^2/R \Rightarrow N = mg \cos \theta + mv^2/R > 0 \text{ always}$$

So **N can never vanish in the lower half**. The only danger zone is the upper half, between 1 and 2 — and once $v_0 \geq \sqrt{5gR}$, N is not zero even at the top, so it is never zero anywhere.

2. Outer track — sliding off a smooth sphere / hump

Now the mass sits on the **outside** of the circle at the top (point 1) and starts sliding from rest, no friction. It follows the circle down to some angle θ , then leaves and follows a parabola. Find θ . Same two tools.



Outer track: released from rest at the top, the mass leaves at point 3 where $N = 0$. Here the radial component of mg , $mg \cos \theta$, is the only inward force.

Energy (1 $\rightarrow$ 3): loss in PE = gain in KE, with $K_1 = 0$ and $h = R - R \cos \theta$:

$$mgh = \frac{1}{2} m v_3^2 \Rightarrow v_3^2 = 2gR(1 - \cos \theta) \quad (1)$$

Force at 3, inward along the radius, with $N = 0$:

$$mg \cos \theta - N = m v_3^2 / R, \quad N = 0 \quad \Rightarrow \quad g \cos \theta = 2g(1 - \cos \theta)$$

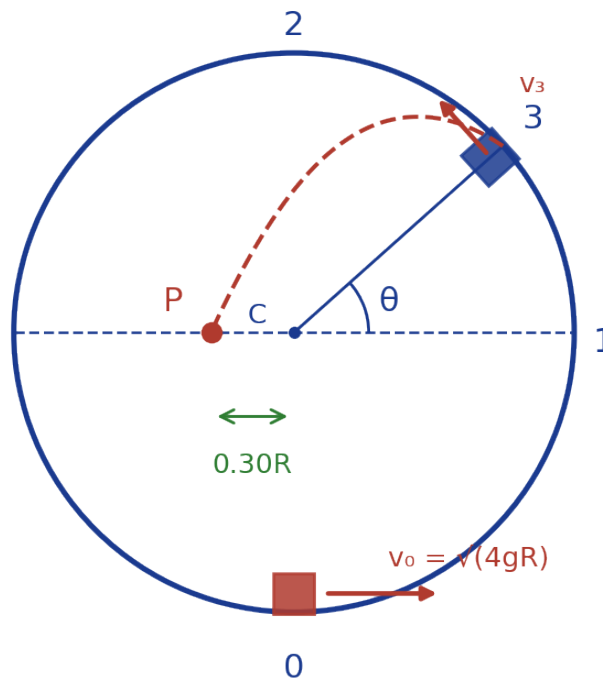
$$\cos \theta = 2/3, \quad \theta = \cos^{-1}(2/3) \approx 48^\circ \quad (\text{a very celebrated result — remember it})$$

At that moment the mass is at height $R \cos \theta = 2R/3$ above the centre and its speed is $v_3^2 = 2gR/3$.

3. Two solved examples — what happens after the mass leaves the track

Same recipe both times: ① find the leaving point (energy + force with $N = 0$) — already done above; ② resolve v_3 along the tangent at that point; ③ treat the rest as a projectile.

Example A — inner track, $v_0 = \sqrt{4gR}$: where does it leave, and where does it cross the horizontal line through C?



Leaves at 3 ($\sin \theta = 2/3$), rises a little, then falls and crosses the horizontal diameter at P, $0.30R$ on the far side of C — inside the circle. Schematic: the gap between the parabola and the track is exaggerated (to scale the apex is only $0.05R$ inside the circle); the mass does not touch the track again until well below the diameter.

Take C as origin, x to the right (the side the mass climbs), y upward.

① **Leaving point** (Case 4): $\sin \theta = 2/3$, $\cos \theta = \sqrt{5}/3$, $v_3^2 = gR \sin \theta = 2gR/3$. Position of 3: $x_3 = R \cos \theta = \sqrt{5}R/3 \approx 0.745R$, $y_3 = R \sin \theta = 2R/3$.

② **Velocity at 3.** Along the tangent, perpendicular to the radius, still moving anticlockwise — so up and to the left, at angle θ to the vertical:

$$v_x = -v_3 \sin \theta = -(2/3) v_3, \quad v_y = +v_3 \cos \theta = (\sqrt{5}/3) v_3, \quad v_y^2 = 10gR/27$$

③ **Projectile until $y = 0$.** $0 = y_3 + v_y t - \frac{1}{2}gt^2$, taking the + root (it rises first, then comes down through the line):

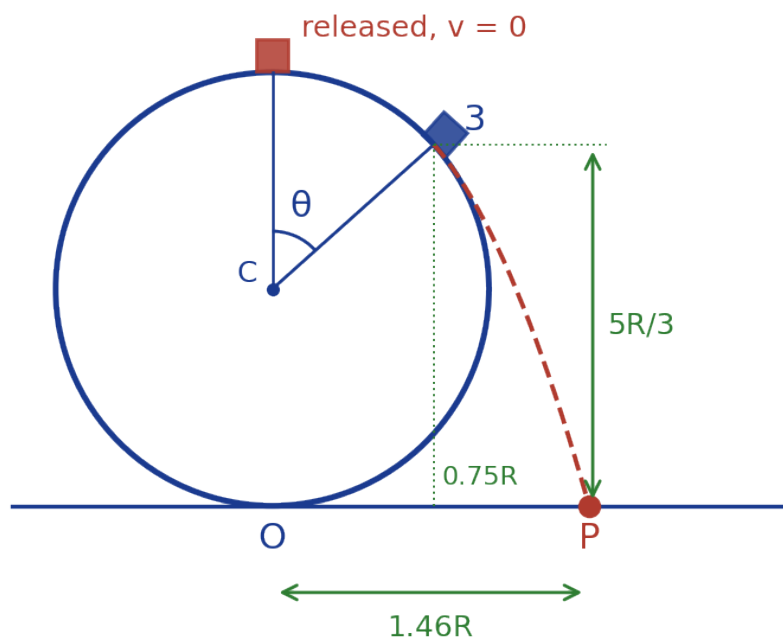
$$t = [v_y + \sqrt{(v_y^2 + 2g y_3)}]/g = [\sqrt{(10gR/27)} + \sqrt{(10gR/27 + 4gR/3)}]/g = (\sqrt{10} + \sqrt{46})/(3\sqrt{3}) \cdot \sqrt{(R/g)} \approx 1.91 \sqrt{(R/g)}$$

$$x_P = x_3 + v_x t = \sqrt{5}R/3 - (2/3)\sqrt{(2gR/3)} \cdot (\sqrt{10} + \sqrt{46})/(3\sqrt{3}) \cdot \sqrt{(R/g)} = R(5\sqrt{5} - 4\sqrt{23})/27 \approx -0.30R$$

Answer. The mass leaves at $\sin^{-1}(2/3) \approx 42^\circ$ above the horizontal on the rising side, crosses the horizontal line through C at **$0.30R$ on the far side of C** (well inside the circle) after **$t \approx 1.91\sqrt{(R/g)}$** — for $R = 1$ m, about

0.61 s — and goes on to strike the lower part of the track. Check that the parabola stays inside the circle: it left tangentially *inward* with $N = 0$, and at its apex it is $\approx 0.95R$ from C.

Example B — sliding off a smooth sphere resting on the ground: where does it land?



Sphere of radius R with lowest point O on the ground. The mass leaves at 3 ($\cos \theta = 2/3$), height $5R/3$, and lands at P , $1.46R$ from O .

① **Leaving point** (Section 2): $\cos \theta = 2/3$, $\sin \theta = \sqrt{5}/3$, $v_3^2 = 2gR(1 - \cos \theta) = 2gR/3$. Height of 3 above the ground: $R + R \cos \theta = 5R/3$; horizontal distance of 3 from the vertical through O : $R \sin \theta = \sqrt{5}R/3 \approx 0.745R$.

② **Velocity at 3.** Tangent to the sphere, i.e. at angle θ below the horizontal:

$$v_x = v_3 \cos \theta = (2/3) v_3, \quad v_y = -v_3 \sin \theta = -(\sqrt{5}/3) v_3 \text{ (downward)}, \quad v_y^2 = 10gR/27$$

③ **Fall to the ground.** $5R/3 = |v_y| t + \frac{1}{2}gt^2$:

$$t = [-|v_y| + \sqrt{(v_y^2 + 10gR/3)}]/g = [-\sqrt{(10gR/27)} + \sqrt{(100gR/27)}]/g = (10 - \sqrt{10})/(3\sqrt{3}) \cdot \sqrt{(R/g)} \approx 1.32 \sqrt{(R/g)}$$

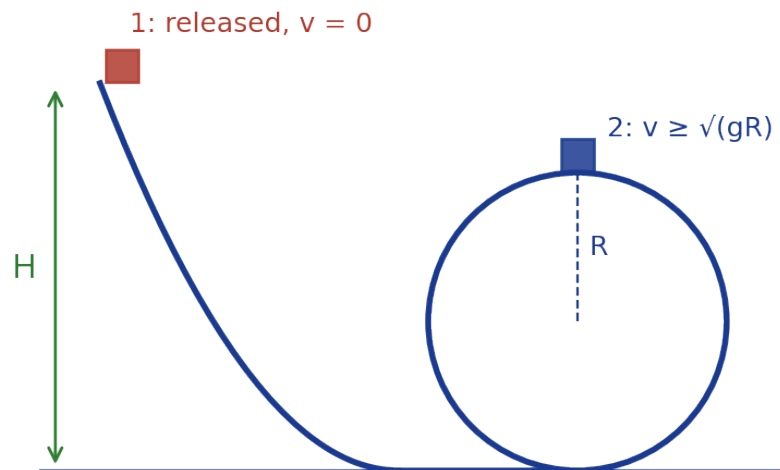
$$v_x t = (2/3)\sqrt{(2gR/3)} \cdot (10 - \sqrt{10})/(3\sqrt{3}) \cdot \sqrt{(R/g)} = (20\sqrt{2} - 4\sqrt{5}) R/27 \approx 0.716 R$$

$$OP = R \sin \theta + v_x t = (5\sqrt{5} + 20\sqrt{2}) R/27 = 5(\sqrt{5} + 4\sqrt{2}) R/27 \approx 1.46 R$$

Answer. It lands $1.46R$ from the bottom contact point, about $1.32\sqrt{(R/g)}$ after leaving the sphere (for $R = 1$ m: 1.46 m, 0.42 s). It never touches the sphere again after leaving — it left tangentially *outward*, and the parabola falls away from the surface.

- The exact surd forms are for completeness. What you should be able to produce in an exam are the numbers: $0.30R$ and $1.91\sqrt{(R/g)}$ in A; $1.46R$ and $1.32\sqrt{(R/g)}$ in B.

4. Loop-the-loop: minimum height on a smooth track



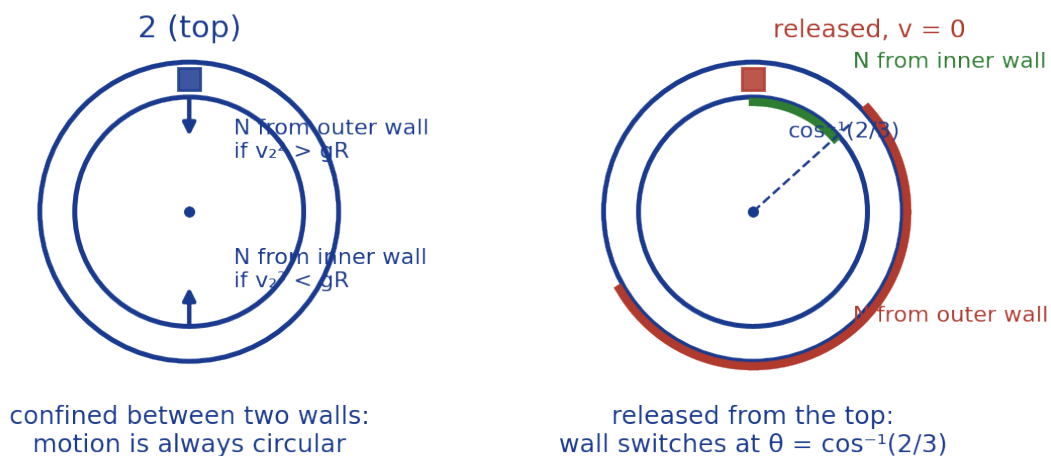
Released from rest at height H on a smooth track that ends in a vertical loop of radius R . To go round, the speed at the top of the loop must be at least $\sqrt{gR}$.

The force equation at the top has already been done: $v_2 \geq \sqrt{gR}$. Now energy between 1 (release) and 2 (top of loop):

$$mg(H - 2R) = \frac{1}{2} m (gR) \Rightarrow H = 2R + R/2 \Rightarrow H_{\min} = 5R/2$$

Variation with friction. If the track is rough and the mass loses energy equal to, say, $mgR/3$ on the way, that much extra height is needed at the start: $H = 5R/2 + R/3 = 17R/6$. In general $mgH = 5mgR/2 + (\text{energy lost})$.

5. Channel track — the mass runs between two circles



Left: with walls on both sides the mass cannot fly off, so the motion is always circular; which wall pushes depends on v^2 .
Right: released from rest at the top, the inner wall pushes until $\theta = \cos^{-1}(2/3)$, the outer wall pushes after that.

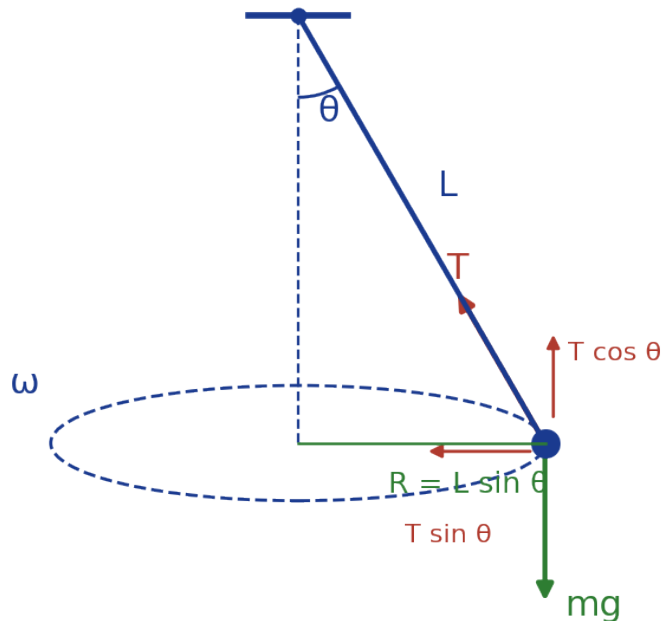
- Parabolic motion is now impossible — the motion is **always circular**. But N can come from either wall.
- At the top: if $v_2^2 > gR$ the mass "wants" to go outward, so **N comes from the outer circle**. If $v_2^2 < gR$ it wants to fall inward but the inner wall stops it, so **N comes from the inner circle**.
- Released from the top of the channel: for θ up to $\cos^{-1}(2/3)$ the inner wall supplies N (as on the outer track). Beyond that the speed has grown too large — on an open track it would have flown off — but now the outer wall holds it in, and N comes from the outer wall for the rest of the circle.

- (Added for completeness.) Given v_0 at the bottom, the mass reaches the top of a channel at all only if $v_2^2 = v_0^2 - 4gR \geq 0$, i.e. $v_0 \geq 2\sqrt{gR}$; below that it oscillates, exactly as on the open inner track — the channel only matters in the upper half.

Sir's remark in class: these are good questions for good-quality JEE Main and average JEE Advanced. Not all Advanced questions are extremely difficult — 20–25% are of this level. Get the basics right and you are through.

6. Conical pendulum

A bob on a string of length L moves in a **horizontal** circle of radius R ; the string sweeps out a cone at angle θ to the vertical. It is exactly like banking of a road with no friction: the tension plays the role of the normal reaction.



Conical pendulum: $T \cos \theta$ balances mg vertically; $T \sin \theta$ supplies the centripetal force; $R = L \sin \theta$.

$$T \cos \theta = mg, \quad T \sin \theta = m v^2 / R = m \omega^2 R, \quad R = L \sin \theta$$

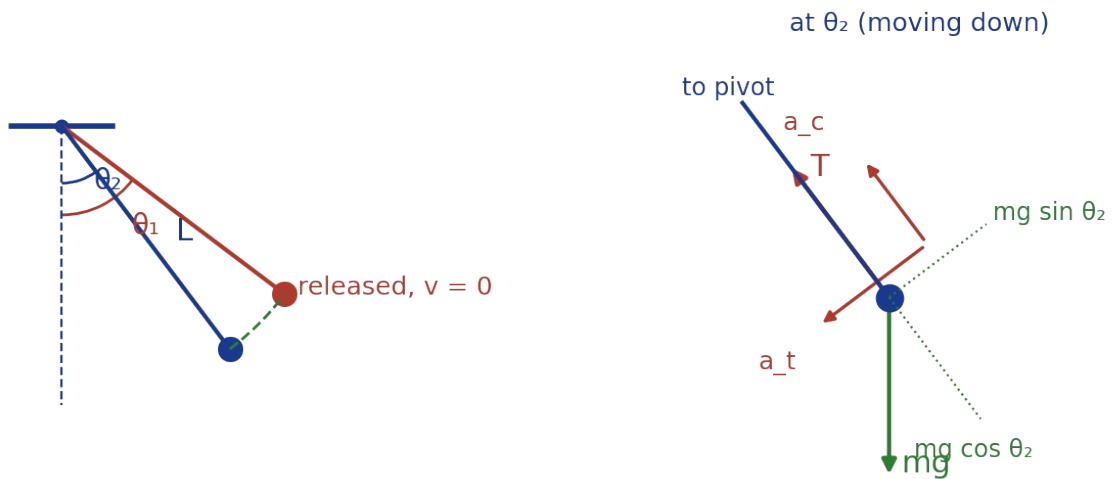
Divide, using ω (the cleaner form):

$$\tan \theta = \omega^2 R / g = \omega^2 L \sin \theta / g \quad \Rightarrow \quad \cos \theta = g / (\omega^2 L) \quad (\text{often asked directly})$$

Using v instead: $\tan \theta = v^2 / (Rg)$, so $v^2 = gR \tan \theta = gL \sin^2 \theta / \cos \theta$ — a clumsier expression, which is why the ω form is worth remembering. Also $T = mg / \cos \theta$, and the period is $T_{\text{period}} = 2\pi / \omega = 2\pi \sqrt{L \cos \theta / g}$.

7. Example — pendulum released from an angle

Question. A bob of mass m on a string of length L is released from rest when the string makes angle θ_1 with the vertical. When the angle has reduced to θ_2 , find (i) the tension, (ii) the acceleration of the bob.



Left: released at θ_1 , the bob swings down through θ_2 . Right: forces and accelerations at θ_2 — T and $mg \cos \theta_2$ along the string, $mg \sin \theta_2$ along the tangent; a_c along the string towards the pivot, a_t along the tangent.

Step 1 — speed at θ_2 (energy). The bob drops through $\Delta h = L(\cos \theta_2 - \cos \theta_1)$:

$$mg \Delta h = \frac{1}{2} m v_1^2 \quad \Rightarrow \quad v_1^2 = 2gL (\cos \theta_2 - \cos \theta_1)$$

Step 2 — tension (force along the string, towards the pivot):

$$T - mg \cos \theta_2 = m v_1^2 / L = 2mg(\cos \theta_2 - \cos \theta_1) \quad \Rightarrow \quad T = mg(3 \cos \theta_2 - 2 \cos \theta_1)$$

Step 3 — acceleration. The bob's acceleration has two components: centripetal (along the string, because it is moving on a circle) and tangential (along the tangent, because its speed is changing).

$$a_c = v_1^2 / L = 2g(\cos \theta_2 - \cos \theta_1), \quad a_t = g \sin \theta_2$$

$$a = \sqrt{(a_c)^2 + (a_t)^2} = g \sqrt{(\sin^2 \theta_2 + 4(\cos \theta_2 - \cos \theta_1)^2)}$$

Direction: if α is the angle that a makes with the string, then since a_c lies along the string, $\tan \alpha = a_t / a_c$. (a_c / a_t would give the angle with the tangent.)

Numbers, to avoid surds: take $\theta_1 = 53^\circ$, $\theta_2 = 37^\circ$ ($\cos 53^\circ = 0.6$, $\cos 37^\circ = 0.8$, $\sin 37^\circ = 0.6$). Then $T = mg(2.4 - 1.2) = 1.2 mg$; $a_c = 2g(0.2) = 0.4g$; $a_t = 0.6g$; $a = g\sqrt{(0.36 + 0.16)} = g\sqrt{0.52} \approx 0.72g$; $\tan \alpha = 0.6/0.4 = 1.5$, $\alpha \approx 56^\circ$ from the string.

*Sir's remark in class: the student summed it up at the end: "force perspective and energy perspective — that is what mechanics is." That is the right instinct, and today's class used exactly those two: the force equation at one point, the energy equation between two points. Keep one more in the same family — the **momentum perspective** (impulse-momentum, conservation of momentum), which handles collisions and explosions. All three are the same law, $F = ma$: used directly (force), integrated over distance (energy), integrated over time (momentum). One law, three ways of using it — plus the constraint equations (string lengths, rolling) that tie a problem together.*

8. Formula sheet — results to remember from today

Situation	Result	Where it comes from
Inner track, exactly reach the side point	$v_0 = \sqrt{2gR}$	energy, $v_1 = 0$
Inner track, $v_0 < \sqrt{2gR}$	$\cos \theta = 1 - v_0^2/2gR$; oscillates	energy
Inner track, minimum speed at top	$v_2 = \sqrt{gR}$	force at top, $N = 0$
Inner track, complete the circle	$v_0 \geq \sqrt{5gR}$	energy $0 \rightarrow 2$ with $v_2 = \sqrt{gR}$
Inner track, $\sqrt{2gR} < v_0 < \sqrt{5gR}$	leaves at $\sin \theta = (v_0^2 - 2gR)/3gR$ (above horizontal)	energy + force, $N = 0$
Inner track, lower half	$N = mg \cos \theta + mv^2/R$, never zero	force
Outer track, released from top	leaves at $\cos \theta = 2/3$	energy + force, $N = 0$
Loop-the-loop, smooth	$H_{\min} = 5R/2$	energy + $v_{\text{top}} = \sqrt{gR}$
Channel track	always circular; wall switches where $v^2 = gR$ at the top / $\cos \theta = 2/3$ from the top	sign of N
Conical pendulum	$\cos \theta = g/(\omega^2 L)$; $T = mg/\cos \theta$	$T \cos \theta = mg$, $T \sin \theta = m\omega^2 R$
Pendulum released at θ_1 , at θ_2	$T = mg(3 \cos \theta_2 - 2 \cos \theta_1)$; $a_c = 2g(\cos \theta_2 - \cos \theta_1)$; $a_t = g \sin \theta_2$	energy + force

- **A mass on a string** behaves exactly like the inner track, with **T** in place of **N**: the string goes slack ($T = 0$) where the track would lose contact. All the inner-track results carry over unchanged.
- **Tension difference** for a body on a string of length R : $T_{\text{bottom}} - T_{\text{top}} = 6mg$, whatever the speed (from the two force equations and one energy equation).

Check yourself before the next class

1. A mass is projected on a smooth inner track with $v_0 = \sqrt{3gR}$. Where does it leave the track? [$\sin \theta = 1/3$, $\theta \approx 19.5^\circ$ above the horizontal]
2. A bob on a string of length R has speed $\sqrt{6gR}$ at the bottom. Find the tension at the top and at the bottom. [$T_{\text{top}} = mg$, $T_{\text{bottom}} = 7mg$]
3. A particle slides from rest off the top of a smooth sphere of radius R . At what height above the centre does it leave, and with what speed? [$2R/3$; $v^2 = 2gR/3$]
4. On a smooth loop-the-loop track a mass is released from $H = 3R$. Find the normal reaction at the top of the loop. [$v_{\text{top}}^2 = 2gR$; $N = mg$]
5. A conical pendulum of length 1 m makes 60° with the vertical. Find ω and the tension ($g = 10 \text{ m/s}^2$). [$\omega^2 = 20$, $\omega \approx 4.5 \text{ rad/s}$; $T = 2mg$]
6. A pendulum is released from the horizontal ($\theta_1 = 90^\circ$). Find T , a_c and a_t at $\theta_2 = 60^\circ$ and at the bottom. [at 60° : $T = 1.5mg$, $a_c = g$, $a_t = 0.87g$; at bottom: $T = 3mg$, $a_c = 2g$, $a_t = 0$]

Flagged for next class

- Projectile follow-ups of the kind in Section 3 with other launch speeds; and the string version of Case 4 ($T = 0$).
- Vertical circle on a string with a peg / with the string cut at the top — same tools, **T** instead of **N**.
- Remaining work–power–energy topics on the list: power, conservation of momentum, centre of mass (rotation is a separate chapter).