Calculus Problems — Class XII Maths

Worked solutions for Shivansh · Integration, Application of Derivatives, Differentiation

Q75. Integration by partial fractions

75. $\int \frac{(x^2 + 1)(x^2 + 4)}{(x^2 + 3)(x^2 - 5)}\, dx$

Step 1. Why we cannot start with partial fractions straight away.

Expand top and bottom — notice that everything is in powers of x^2:

$$(x^2 + 1)(x^2 + 4) = x^4 + 5x^2 + 4$$

$$(x^2 + 3)(x^2 - 5) = x^4 - 2x^2 - 15$$

Top and bottom have the **same degree** (4). Partial fractions need a **proper** fraction, so divide first.

Step 2. Divide to get a proper fraction.

$$x^4 + 5x^2 + 4 = (x^4 - 2x^2 - 15) + (7x^2 + 19)$$

Check: $-2x^2 + 7x^2 = 5x^2$ and $-15 + 19 = 4$. ✓ Hence

$$\frac{(x^2+1)(x^2+4)}{(x^2+3)(x^2-5)} = 1 + \frac{7x^2 + 19}{(x^2+3)(x^2-5)}$$

Step 3. Partial fractions.

Everything is in x^2, so constant numerators A and B are enough (no x-terms needed):

$$\frac{7x^2 + 19}{(x^2 + 3)(x^2 - 5)} = \frac{A}{x^2 + 3} + \frac{B}{x^2 - 5}$$

$$7x^2 + 19 = A(x^2 - 5) + B(x^2 + 3)$$

Quickest way — put in the values of x^2 that kill one bracket each:

$$x^2 = 5: \quad 35 + 19 = 8B \quad \Rightarrow \quad B = \frac{27}{4}$$

$$x^2 = -3: \quad -21 + 19 = -8A \quad \Rightarrow \quad A = \frac{1}{4}$$

$$\text{Integrand} = 1 + \frac{1}{4(x^2 + 3)} + \frac{27}{4(x^2 - 5)}$$

Step 4. Integrate term by term.

Recall: $\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1}\frac{x}{a} + C$ and $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log\left|\frac{x - a}{x + a}\right| + C$

Here $a = \sqrt{3}$ and $a = \sqrt{5}$ (since $5 = (\sqrt{5}\,)^2$). Multiply out $1/4 \times 1/\sqrt{3}$ and $27/4 \times 1/(2\sqrt{5}\,)$:

$$\boxed{I = x + \frac{1}{4\sqrt{3}} \tan^{-1}\frac{x}{\sqrt{3}} + \frac{27}{8\sqrt{5}} \log\left|\frac{x - \sqrt{5}}{x + \sqrt{5}}\right| + C}$$

*Two things that are easy to forget in the exam: the **modulus** inside the log (the fraction is negative when $-\sqrt{5} < x < \sqrt{5}$) and the constant **+ C**. Both are worth marks.*

Check: differentiating the answer gives back $1 + \frac{1}{4(x^2+3)} + \frac{27}{4(x^2-5)}$, the split of Step 3. ✓

Q7. Rate of change — draining a pool

7. A swimming pool is to be drained for cleaning. If L represents the number of litres of water in the pool t seconds after the pool has been plugged off to drain and $L = 200(10 - t)^2$. How fast is the water running out at the end of 5 seconds? What is the average rate at which the water flows out during first 5 seconds?

Understand the model first.

$$L(t) = 200(10 - t)^2 \text{ litres}, \quad 0 \le t \le 10$$

At $t = 0$, $L = 200 \times 100 = 20\,000$ litres (the full pool). At $t = 10$, $L = 0$ — the pool is empty after 10 s. L keeps **decreasing**, so $\frac{dL}{dt}$ must come out negative.

(i) Instantaneous rate at $t = 5$ s.

Differentiate using the chain rule (the inner function is $10 - t$, whose derivative is -1):

$$\frac{dL}{dt} = 200 \times 2(10 - t) \times (-1) = -400(10 - t) \quad \text{litres/s}$$

The negative sign says the amount of water is falling. The **rate at which water runs out** is the size of this:

$$\text{Rate out} = -\frac{dL}{dt} = 400(10 - t)$$

$$\text{At } t = 5: \quad 400 \times (10 - 5) = 2000 \quad \text{litres/s}$$

$\boxed{\text{Water is running out at 2000 litres per second at } t = 5 \text{ s}}$

(ii) Average rate over the first 5 seconds.

Average rate $= \frac{\text{total change in } L}{\text{time taken}}$ — it does not need the derivative at all.

$L(0) = 200(10)^2 = 20\,000$ litres, $\quad L(5) = 200(10 - 5)^2 = 200 \times 25 = 5000$ litres

Water drained in first 5 s $= 20\,000 - 5000 = 15\,000$ litres

$$\text{Average rate} = \frac{15\,000}{5} = 3000 \quad \text{litres/s}$$

$\boxed{\text{Average rate of outflow during the first 5 s} = 3000 \text{ litres per second}}$

Why the two answers differ. The pool drains fastest at the start (rate out = 4000 L/s at $t = 0$) and slows down as the water level, and hence the pressure, drops. The average over 0–5 s (3000 L/s) sits between the starting value 4000 and the value 2000 at $t = 5$, as it should. Notice the rate out is linear in t, so the average of a linear function over an interval is just the mean of its end values: $(4000 + 2000)/2 = 3000$. ✓

Side note on the rough work: $\frac{d^2L}{dt^2} = +400$ for L itself (the graph of L is a parabola opening upwards). It is the **drained volume** $V = 20\,000 - L$ whose second derivative is -400. Either way, the outflow rate $400(10 - t)$ decreases steadily by 400 L/s every second.

Q52. *Differentiate an inverse-trig expression* (2018)

52. Differentiate $\tan^{-1}\left(\dfrac{1 + \cos x}{\sin x}\right)$ with respect to x.

***Method** 1 (**simplify first — the intended method**).*

Use the half-angle identities:

Recall: $1 + \cos x = 2\cos^2\dfrac{x}{2}$ $\sin x = 2\sin\dfrac{x}{2}\cos\dfrac{x}{2}$

$$\frac{1 + \cos x}{\sin x} = \frac{2\cos^2(x/2)}{2\sin(x/2)\cos(x/2)} = \frac{\cos(x/2)}{\sin(x/2)} = \cot\frac{x}{2}$$

To get rid of $\tan^{-1}$ *we want a* ***tan****, not a cot. Use* $\cot\theta = \tan\left(\dfrac{\pi}{2} - \theta\right)$:

$$y = \tan^{-1}\left[\tan\left(\frac{\pi}{2} - \frac{x}{2}\right)\right]$$

Now $\tan^{-1}(\tan\theta) = \theta$ *provided* θ *lies in* $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$. *For* $0 < x < \pi$ *we have* $0 < \dfrac{\pi}{2} - \dfrac{x}{2} < \dfrac{\pi}{2}$, *so this is fine and*

$$y = \frac{\pi}{2} - \frac{x}{2}$$

$$\boxed{\frac{dy}{dx} = -\frac{1}{2}}$$

***Method** 2 (**implicit differentiation — a good cross-check**).*

Let $\tan y = \dfrac{1 + \cos x}{\sin x}$. *Differentiate both sides with respect to x, using the quotient rule on the right:*

$$\sec^2 y\,\frac{dy}{dx} = \frac{(-\sin x)(\sin x) - (1 + \cos x)(\cos x)}{\sin^2 x}$$

$$= \frac{-\sin^2 x - \cos x - \cos^2 x}{\sin^2 x} = \frac{-(1 + \cos x)}{\sin^2 x}$$

(using $\sin^2 x + \cos^2 x = 1$*). Now replace* $\sec^2 y$ *by* $1 + \tan^2 y$:

$$\sec^2 y = 1 + \frac{(1 + \cos x)^2}{\sin^2 x} = \frac{\sin^2 x + 1 + 2\cos x + \cos^2 x}{\sin^2 x} = \frac{2(1 + \cos x)}{\sin^2 x}$$

Substitute back — the $\sin^2 x$ *cancels from both sides:*

$$2(1 + \cos x)\frac{dy}{dx} = -(1 + \cos x) \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{1}{2} \quad ✓$$

Both methods agree. In the exam, Method 1 is faster (2–3 lines) and is what the marking scheme expects; Method 2 is longer but works even when you cannot spot the identity. It is worth knowing both.

***Remember the pattern**:* $(1 + \cos x)/\sin x \to \cot(x/2)$; $\sin x/(1 + \cos x) \to \tan(x/2)$; $(1 - \cos x)/\sin x \to \tan(x/2)$. *Half-angles turn all of these into a single ratio.*

Working without a calculator

Nothing in these three questions needs a decimal answer, but if a numerical value of the integral is ever asked for, keep these handy:

$$\sqrt{3} \approx 1.732, \quad \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \approx 0.577, \quad \sqrt{5} \approx 2.236, \quad \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5} \approx 0.447$$

$$\frac{1}{4\sqrt{3}} = \frac{0.577}{4} \approx 0.144 \qquad \frac{27}{8\sqrt{5}} = \frac{27 \times 0.447}{8} = \frac{12.07}{8} \approx 1.509$$

Partial-fraction shortcuts.

1. Compare degrees before anything else. If deg(top) ≥ deg(bottom), divide first.
2. When every power of x is even, treat x^2 as the variable. The quadratic factors then behave like linear ones and need only constant numerators.
3. To find the constants, substitute the value of x^2 that makes each factor zero (cover-up method) — one line per constant, no simultaneous equations.
4. Write the answer in the form the standard integrals use: $x^2 + 3 = x^2 + (\sqrt{3})^2$, $x^2 - 5 = x^2 - (\sqrt{5})^2$.

Rates of change.

Instantaneous rate = derivative at that instant. Average rate over an interval = $\dfrac{\text{change in the quantity}}{\text{length of the interval}}$. A negative derivative means the quantity is decreasing; "rate at which it runs out" is the magnitude.